The FFLAS-FFPACK and LinBox libraries

OpenDreamKit Software presentation

Clément Pernet & the LinBox group

September 2, 2015

Exact linear algebra

Matrices can be

Dense: store all coefficients

Sparse: store the non-zero coefficients only

Black-box: no access to the storage, only *apply* to a vector

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Coefficient domains:

Word size:
- integers with a priori bounds
- $\mathbb{Z}/p\mathbb{Z}$ for p of ≈ 32 bits

Multi-precision: $\mathbb{Z}/p\mathbb{Z}$ for p of $\approx 100, 200, 1000, 2000, \dots$ bits

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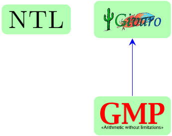
Requires genericity.

Software stack for exact linear algebra

Arithmetic

GMP «Arithmetic without limitations», MPIR: multiprecision integers and rationals

Givaro, NTL: finite fields and polynomials



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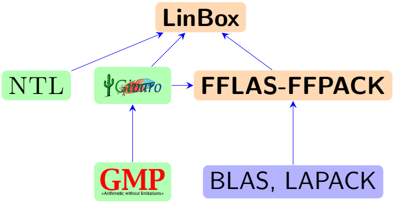
BLAS: Basic Linear Algebra Subroutines (floating point)

FFLAS-FFPACK: Basic Exact Linear Algebra over $\mathbb{Z}/p\mathbb{Z}$,

LinBox: Linear Algebra over $\mathbb{Z}, \mathbb{Z}/p\mathbb{Z}$ and $K[X]$



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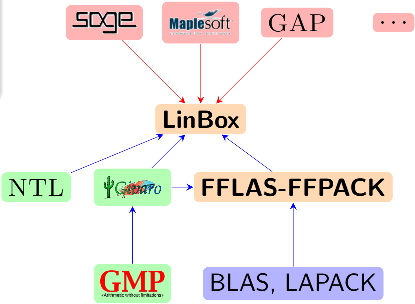

Outline

The LinBox project

- International collaboration: Canada, USA, France
- Strongly generic `C++` code, focus on efficiency
- Free software (LGPL 2.1+)
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Milestones

1998	First design: Black box and sparse matrices
2003	Dense linear algebra using BLAS $\rightsquigarrow$ FFLAS-FFPACK
2005	LinBox-1.0
2008	Integration in Sage
2012-..	Parallelization
2014	SIMD & Sparse BLAS in FFLAS-FFPACK (Brice's talk)

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$\rightsquigarrow$

- Iterative methods
- No access to coefficients, trace, no elimination
- Matrix **multiplication** $\Rightarrow$ Black-box **composition**

Example: blackbox composition

```
template <class Mat1, class Mat2>
class Compose {
  protected:
    Mat1 _A;
    Mat2 _B;
  public:
    Compose(Mat1& A, Mat2& B) : _A(A), _B(B) {}

    template<class InVec, class OutVec>
    OutVec& apply (const InVec& x) {
      return _A.apply(_B.apply(x));
    }
};
```

Black box linear algebra

Matrix-Vector Product: building block,
$\rightsquigarrow$ costs $E(n)$

Minimal polynomial: [Wiedemann 86]
$\rightsquigarrow$ iterative Krylov/Lanczos methods
$\rightsquigarrow O(nE(n) + n^2)$

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Characteristic Poly.: [Dumas P. Saunders 09]
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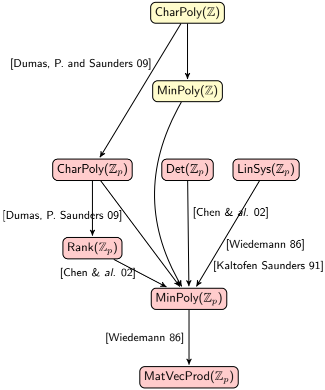

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Reductions: linear algebra's arithmetic complexity

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Matrix Product

[Strassen 69]:	$O(n^{2.807})$
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[Schönhage 81]	$O(n^{2.52})$
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[Coppersmith, Winograd 90]	$O(n^{2.375})$
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[Le Gall 14]	$O(n^{2.3728639})$

$\rightsquigarrow \mathsf{MM}(n) = O(n^\omega)$

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$\rightsquigarrow$ $\mathsf{MM}(n) = O(n^\omega)$

Other operations

[Strassen 69]: Inverse in $O(n^\omega)$

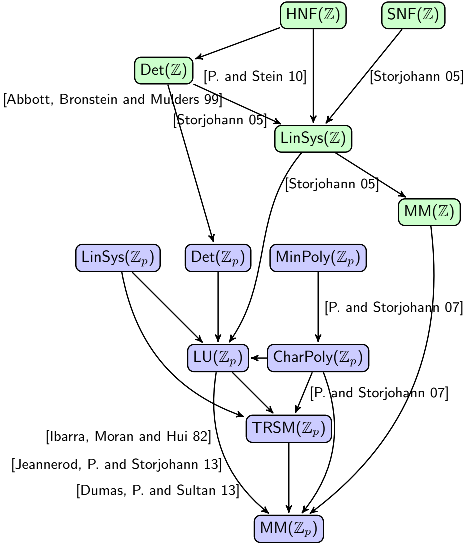
[Schönhage 72]: QR in $O(n^\omega)$

[Bunch, Hopcroft 74]: LU in $O(n^\omega)$

[Ibarra & *al.* 82]: Rank in $O(n^\omega)$

[Keller-Gehrig 85]: CharPoly in $O(n^\omega \log n)$

Reductions


HNF($\mathbb{Z}$)
SNF($\mathbb{Z}$)
Det($\mathbb{Z}$)
[P. and Stein 10]
[Storjohann 05]
[Abbott, Bronstein and Mulders 99]
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LinSys($\mathbb{Z}$)
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[Ibarra, Moran and Hui 82]
[Jeannerod, P. and Storjohann 13]
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- ✓ building blocks **in theory** happen to be the most efficient routines **in practice**

$\rightsquigarrow$ reduction trees are still relevant

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Roadmap

1. Tune building blocks (MatMul)
2. Improve existing reductions (LU, Echelon)
   - ▷ leading constants
   - ▷ memory footprint
3. Produce new reduction schemes (CharPoly)

Matrix Multiplication over $\mathbb{Z}/p\mathbb{Z}$

Ingedients [Dumas, Gautier and P. 02]

- Compute over $\mathbb{Z}$ and delay modular reductions

$\rightsquigarrow$ $k\left(\frac{p-1}{2}\right)^2 < 2^{\text{mantissa}}$

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- Cache optimizations

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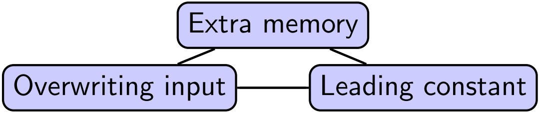
with memory efficient schedules [Boyer, Dumas, P. and Zhou 09]

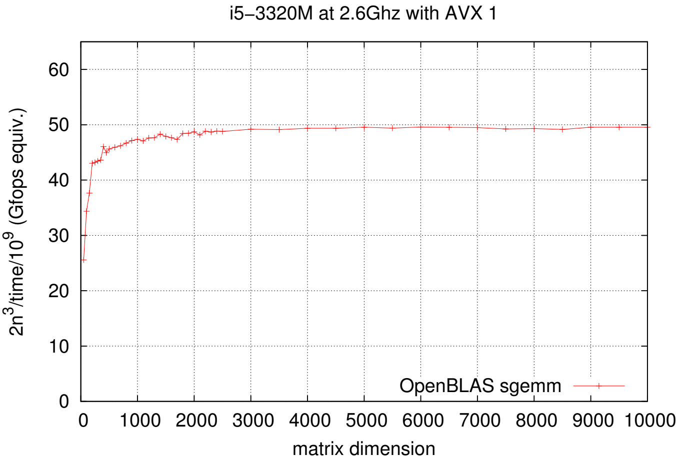
Tradeoffs:



Fully in-place in

$7.2n^{2.807} + \dots$

Sequential Matrix Multiplication



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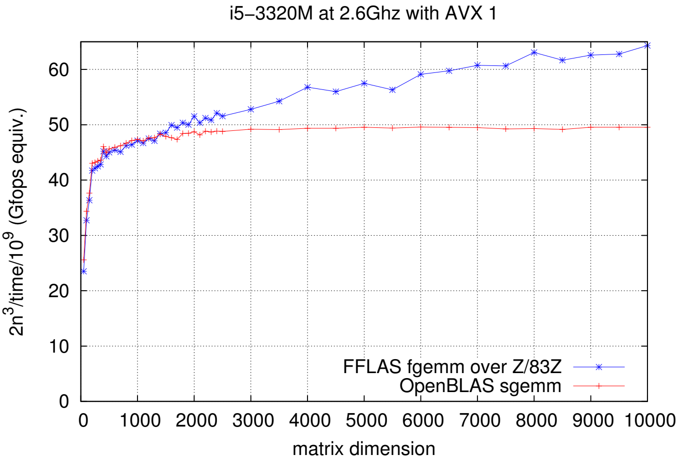

$p = 83$, $\rightsquigarrow$ 1 mod / 10000 mul.

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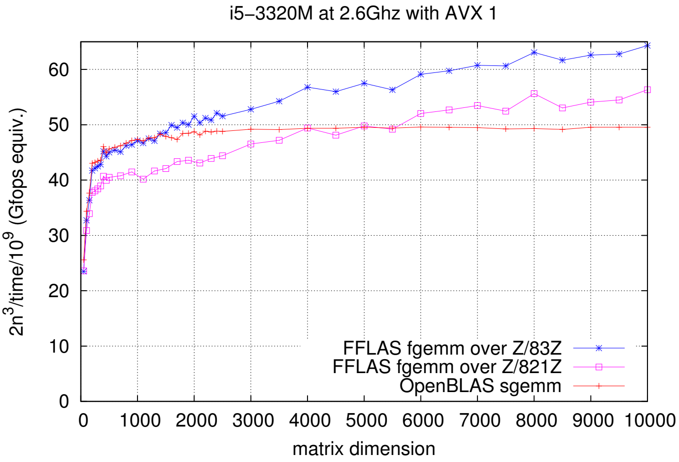

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$p = 821$, $\rightsquigarrow$ 1 mod / 100 mul.

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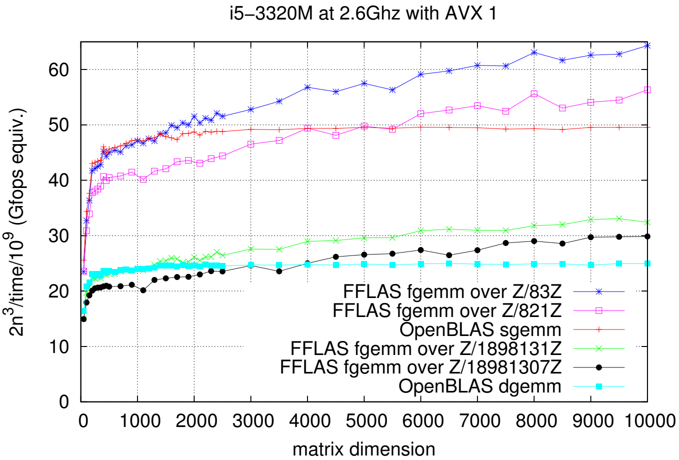

$p = 83$, $\rightsquigarrow$ 1 mod / 10000 mul. $p = 1898131$, $\rightsquigarrow$ 1 mod / 10000 mul.

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Other routines

LU decomposition

- Block recursive algorithm $\rightsquigarrow$ reduces to MatMul $\rightsquigarrow$ $O(n^\omega)$

n	1000	5000	10000	15000	20000
`LAPACK-dgetrf`	**0.024s**	**2.01s**	**14.88s**	48.78s	113.66
`fflas-ffpack`	0.058s	2.46s	16.08s	**47.47s**	**105.96s**

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- A new reduction to matrix multiplication in $O(n^\omega)$.

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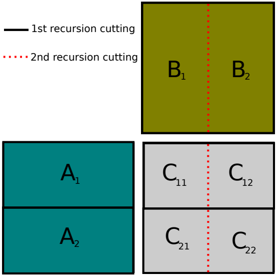
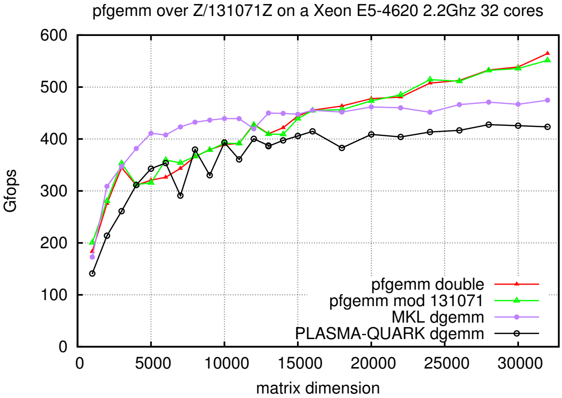
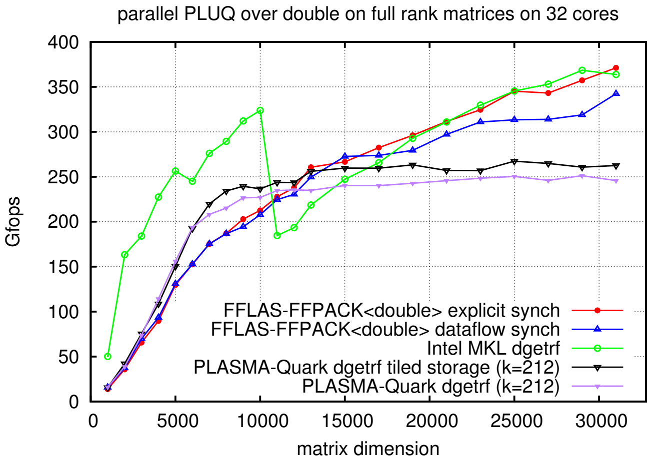
Parallel matrix multiplication





Gaussian elimination



Thank You.